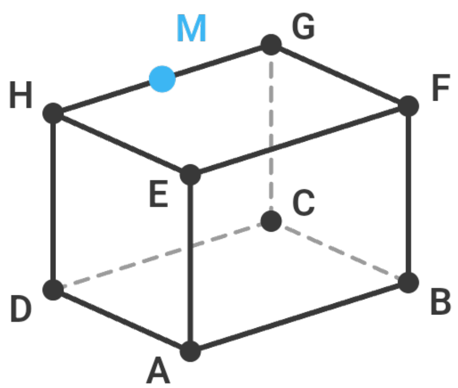
Secondary 5 - Mathematics

Angles and Distances in 3-D Solids

Name: ____________________________

Class: ____________________________

1. In the figure, $ABCDEFGH$ is a rectangular block in which E, F, G and H are vertically above A, B, C and D respectively. It is given that $AB = 12\,\text{cm}$, $BC = 9\,\text{cm}$ and $AE = 8\,\text{cm}$. M is the mid-point of GH.



Find the angle between AM and the plane $ABCD$, correct to the nearest degree.

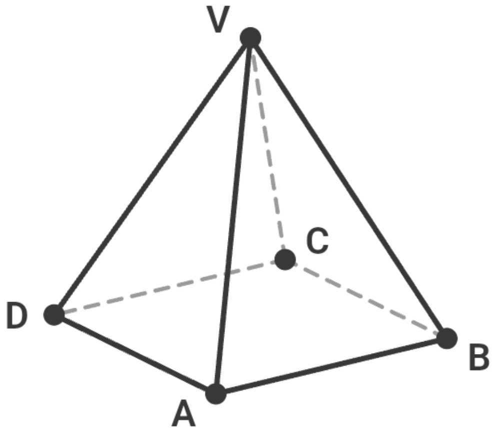
A. 28°

B. 36°

C. 42°

D. 53°

2. The figure shows a right pyramid $VABCD$ whose base $ABCD$ is a square, where $AB = 10\,\text{cm}$ and $VA = 13\,\text{cm}$.



Let θ be the dihedral angle between the face VAB and the face VBC measured inside the pyramid along their common edge VB. Then $\cos\theta =$

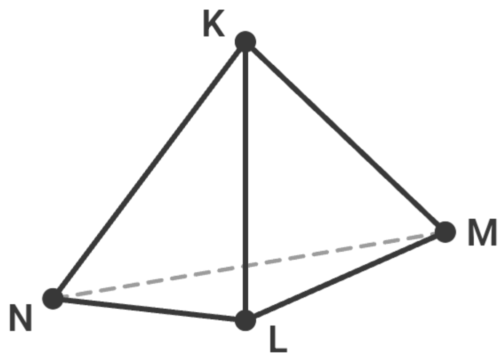
A. $-\dfrac{25}{144}$

B. $\dfrac{25}{144}$

C. $-\dfrac{119}{169}$

D. $-\dfrac{119}{144}$

3. In the figure, the base LMN of the tetrahedron $KLMN$ lies on the horizontal ground and K is vertically above L. It is given that $\angle KML = 40^\circ$, $\angle KNL = 55^\circ$ and $\angle MLN = 105^\circ$.



Find $\angle MKN$, correct to the nearest degree.

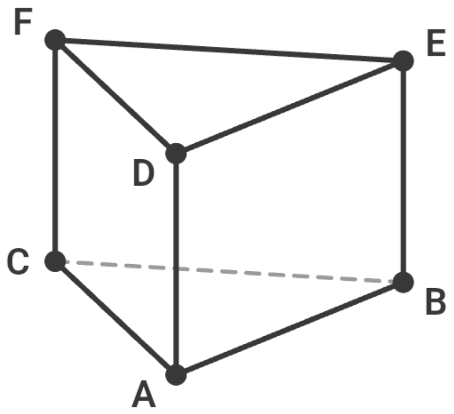
A. 50°

B. 66°

C. 74°

D. 105°

4. The figure shows a right prism $ABCDEF$ standing on horizontal ground, where D, E and F are vertically above A, B and C respectively. The base ABC is an equilateral triangle of side $6\,\mathrm{cm}$ and $AD = 4\,\mathrm{cm}$.



Find the distance from A to the plane DBC, correct to 3 significant figures.

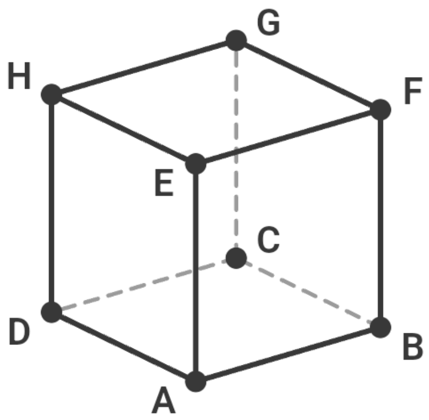
A. $3.17\,\mathrm{cm}$

B. $3.33\,\mathrm{cm}$

C. $4.00\,\mathrm{cm}$

D. $5.20\,\mathrm{cm}$

5. In the figure, $ABCDEFGH$ is a cube in which E, F, G and H are vertically above A, B, C and D respectively.



Let α be the angle between AG and the plane $ABCD$, β be the angle between the plane BDE and the plane $ABCD$, and γ be the angle between AH and the plane $ABCD$. Which of the following is true?

A. $\alpha < \gamma < \beta$

B. $\gamma < \alpha < \beta$

C. $\alpha < \beta < \gamma$

D. $\gamma < \beta < \alpha$

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Answers

1 **B**

> To find the angle between a line and a plane, project the line onto the plane.
> Let N be the point of the base $ABCD$ lying vertically below M. As M is the mid-point of GH, N is the mid-point of CD, and MN is perpendicular to the base with $MN = AE = 8\,\mathrm{cm}$.
> Hence AN is the projection of AM on the plane $ABCD$, and the required angle is $\angle MAN$.
> Step 1: find AN. Going from A to N inside the base we move $6\,\mathrm{cm}$ along the direction of AB (half of $12\,\mathrm{cm}$) and $9\,\mathrm{cm}$ along the direction of AD. By Pythagoras' theorem,
> $AN = \sqrt{6^2 + 9^2} = \sqrt{117}\,\mathrm{cm} \approx 10.8167\,\mathrm{cm}$
> Step 2: use the right-angled triangle AMN (right angle at N):
> $\tan \angle MAN = \dfrac{MN}{AN} = \dfrac{8}{\sqrt{117}} \approx 0.73959$
> $\angle MAN \approx 36.4866^\circ \approx 36^\circ$
> A common slip is to use AB or the base diagonal AC instead of AN; always project the actual end point of the line.

2 A

The two faces meet along VB, so we need two lines through the same point of VB, one lying in each face, both perpendicular to VB.
Let M be the foot of the perpendicular from A to VB. Since $VA = VC = 13\,\text{cm}$ and $AB = CB = 10\,\text{cm}$, triangles VAB and VCB are congruent, so the perpendicular from C to VB meets VB at the same point M. Hence the interior dihedral angle is $\theta = \angle AMC$.
Step 1: find AM. In the isosceles triangle VAB, the height from V to AB is $\sqrt{13^2 - 5^2} = 12\,\text{cm}$, so

$$\text{area of } \triangle VAB = \tfrac{1}{2}(10)(12) = 60\,\text{cm}^2$$

Taking $VB = 13\,\text{cm}$ as the base,

$\tfrac{1}{2}(13)(AM) = 60$, so $AM = \dfrac{120}{13}\,\text{cm} = CM$.

Step 2: find AC. $AC = \sqrt{10^2 + 10^2} = 10\sqrt{2}\,\text{cm}$, so $AC^2 = 200$.
Step 3: cosine rule in $\triangle AMC$.

$$\begin{aligned}
\cos\theta &= \frac{AM^2 + CM^2 - AC^2}{2\,AM \cdot CM} \\
&= \frac{2\left(\frac{14400}{169}\right) - 200}{2\left(\frac{14400}{169}\right)} \\
&= 1 - \frac{200 \times 169}{28800} \\
&= 1 - \frac{169}{144} = -\frac{25}{144}
\end{aligned}$$

The negative value tells you the interior dihedral angle is obtuse (about 100°), which is reasonable for two adjacent faces of a fairly wide pyramid.

3 **B**

Since KL is vertical while LM and LN lie on the horizontal ground, $KL \perp LM$ and $KL \perp LN$. So triangles KLM and KLN are right-angled at L.
Only angles are asked, so let $KL = 10$ for convenience.
Step 1: the two right-angled triangles.

$$LM = \frac{10}{\tan 40^\circ} \approx 11.918, \quad KM = \frac{10}{\sin 40^\circ} \approx 15.557$$

$$LN = \frac{10}{\tan 55^\circ} \approx 7.002, \quad KN = \frac{10}{\sin 55^\circ} \approx 12.208$$

Step 2: cosine rule in the horizontal triangle LMN.

$$\begin{aligned} MN^2 &= 11.918^2 + 7.002^2 - 2(11.918)(7.002)\cos 105^\circ \\ &\approx 142.04 + 49.03 + 43.19 \\ &\approx 234.26 \end{aligned}$$

Step 3: cosine rule in $\triangle KMN$.

$$\begin{aligned} \cos\angle MKN &= \frac{KM^2 + KN^2 - MN^2}{2\,KM \cdot KN} \\ &\approx \frac{242.02 + 149.03 - 234.26}{2(15.557)(12.208)} \\ &\approx \frac{156.79}{379.84} \approx 0.4128 \end{aligned}$$

$\angle MKN \approx 65.6^\circ \approx 66^\circ$
Note that $\angle MKN$ is not the same as $\angle MLN$. For this particular configuration, $\angle MKN \approx 65.6^\circ < 105^\circ$, so the angle must be computed, never copied from the ground angle.

4 **A**

The key idea is to find a plane through A that is perpendicular to the plane DBC. Let M be the mid-point of BC. Since $\triangle ABC$ is equilateral, $AM \perp BC$. Also AD is vertical while BC is horizontal, so $AD \perp BC$. Hence BC is perpendicular to the plane ADM, so the plane DBC is perpendicular to the plane ADM, and these two planes meet along DM.
Therefore the required distance is the perpendicular distance from A to the line DM inside the right-angled triangle ADM.
Step 1: find AM.

$$AM = 6\sin 60^\circ = 3\sqrt{3}\,\mathrm{cm} \approx 5.196\,\mathrm{cm}$$

Step 2: find DM (right angle at A, since AD is vertical and AM is horizontal):

$$DM = \sqrt{4^2 + (3\sqrt{3})^2} = \sqrt{16+27} = \sqrt{43}\,\mathrm{cm}$$

Step 3: use the area of $\triangle ADM$ in two ways. Let the required distance be d.

$$\begin{aligned} \tfrac{1}{2}(DM)(d) &= \tfrac{1}{2}(AD)(AM) \\ d &= \frac{4\times 3\sqrt{3}}{\sqrt{43}} \\ &= \frac{12\sqrt{3}}{\sqrt{43}} \approx 3.17\,\mathrm{cm} \end{aligned}$$

Remember that the distance from a point to a plane is measured along a perpendicular, so you must use the perpendicular height in $\triangle ADM$, not an edge such as AD or AM.

5 **A**

Let the side of the cube be 6, then work out the three angles one by one.
α: G is vertically above C, so AC is the projection of AG on the base and $GC \perp$ plane $ABCD$.

$AC = \sqrt{6^2+6^2} = 6\sqrt{2}$, so $\tan\alpha = \dfrac{6}{6\sqrt{2}} = \dfrac{1}{\sqrt{2}}$, giving $\alpha \approx 35.3^\circ$.

γ: H is vertically above D, so AD is the projection of AH.

$\tan\gamma = \dfrac{6}{6} = 1$, giving $\gamma = 45^\circ$.

β: The planes BDE and $ABCD$ meet along BD. Let O be the mid-point of BD. In the square base $AO \perp BD$, and since AE is vertical, the theorem of three perpendiculars gives $EO \perp BD$. Hence $\beta = \angle EOA$.

$AO = \frac{1}{2}(6\sqrt{2}) = 3\sqrt{2}$, so $\tan\beta = \dfrac{6}{3\sqrt{2}} = \sqrt{2}$, giving $\beta \approx 54.7^\circ$.

Comparing: $35.3^\circ < 45^\circ < 54.7^\circ$, so $\alpha < \gamma < \beta$.
Notice the pattern: with the same vertical rise, the longer the horizontal projection, the smaller the angle, which is why the space diagonal AG gives the smallest angle here.